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Generalized metric spaces with algebraic structures

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ABSTRACT

We discuss generalized metrizable properties on paratopological groups and topological groups. It is proved in this paper that a first-countable paratopological group which is a β-space is developable; and we construct a Hausdorff, separable, non-metrizable paratopological group which is developable. We consider paratopological (topological) groups determined by a point-countable first-countable subspaces and give partial answers to Arhangel'skii's conjecture; Nogura–Shakhmatov–Tanaka's question (Nogura et al., 1993 [23]). We also give a negative answer to a question in Cao et al. (in press) [10]. Finally, remainders of topological groups and paratopological groups are discussed and Arhangel'skii's Theorem (Arhangel'skii, 2007 [3]) is improved.

1. Introduction

This paper is devoted to discussing the generalized metrizable properties on topological algebra. As is known to all, every first-countable topological group is metrizable. However, this does not hold for paratopological group. The Sorgenfrey line [11, Example 1.2.2] with the usual addition is a first-countable paratopological group but not metrizable. More even, we can see in Example 2.1 that a developable paratopological group is unnecessary to be metrizable. We prove that every paratopological group with a left-invariant symmetric is metrizable. Also we discuss the following questions:

Question 1.1 *(Arhangel'skii conjecture).* S_{ω_1} cannot be embedded into a sequential topological group.

Question 1.2. ([19, Problem 15]) Let G be a topological group which is a k-space with a σ-compact-finite k-network, or a space with a point-countable determining cover by metric spaces. Is G paracompact (or meta-Lindelöf)?

Question 1.3. ([23, Question 3.10]) Let G be a topological group determined by a point-countable cover consisting of bisequential spaces. If G is an A-space [21], is it metrizable?

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In [19], a stronger version of Question 1.3 is asked.

Question 1.4. ([19, Problem 16(b)]) Let G be a topological group determining by a point-countable first-countable subsets. If G is an A-space, is G metrizable?

Question 1.5. ([10, Question 4.3]) Is a symmetrizable, Baire, semitopological topological group a topological group?

We prove that a sequential topological group with a point-countable k-network is metrizable or a topological sum of cosmic subspaces. So Question 1.1 is true if G has a point-countable k-network. It is known that a space having a σ-compact-finite k-network or determining by a point-countable metric subsets has a point-countable k-network, and note that k-space with a point-countable k-network is a sequential space [13], hence the answer for Question 1.2 is positive. We also prove that a paratopological group determined by a point-countable cover consisting of first-countable subsets is first-countable if it contains no closed copy of S_ω. It gives an affirmative answer to Question 1.4 (note: an A-space contains no closed copy of S_ω [20]) and a partial answer to Question 1.3. We present a separable, semi-metric, semitopological group with Baire property that is not a paratopological group, which give a negative answer to Question 1.5. In the last section, we discuss remainders of (para-)topological groups and slightly improve Arhangel'skii's Theorem [3].

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ denotes the set of all real, rational and natural numbers. e denotes the neutral element of a group. Reader may refer to [11,12] for notations and terminology not explicitly given here.

2. First-countable paratopological groups

All spaces in this section are Hausdorff unless stated otherwise.

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product mappings of $G \times G$ into G is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product mappings of $G \times G$ into G is jointly continuous. A *semitopological group* G is a group G with a topology such that the product mappings of $G \times G$ into G is separately continuous. In this section, we shall show that every first-countable paratopological group is quasi-metrizable.

A function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a *quasi-metric* (*non-Archimedean quasi-metric*) [12] on the set X if for each $x, y, z \in X$, (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, z) \leqslant d(x, y) + d(y, z)$ ($d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$). A topological space X is said to be *quasi-metrizable* (*non-Archimedean quasi-metrizable*) [12] if there is a quasi-metric (non-Archimedean quasi-metric) on X such that $\{B(x, \varepsilon)\colon \varepsilon > 0\}$ forms a local base at each $x \in X$, where $B(x, \varepsilon) = \{y \in X\colon d(x, y) < \varepsilon\}$. Every non-Archimedean quasi-metrizable space is quasi-metrizable. The Sorgenfrey line [11, Example 1.2.2] is a non-Archimedean quasi-metrizable space.

The following proposition was proved by Ravsky [25], we present a new proof here.

By using the same proof of [12, Theorem 10.2], we have the following.

Lemma 2.1. *A Hausdorff space (X, τ) is quasi-metrizable if and only if there is a function $g : \omega \times X \to \tau$ such that* (i) $\{g(n, x)\colon n \in \omega\}$ *is a local base at* x; (ii) $y \in g(n + 1, x) \Rightarrow g(n + 1, y) \subset g(n, x)$.

Proposition 2.1. *Every Hausdorff first-countable paratopological group is quasi-metrizable.*

Proof. Suppose (X, τ) is a first-countable paratopological group. Let $\{V_n\colon n \in \omega\}$ be a countable local base at the neutral element e such that $V_{n+1}^2 \subset V_n$. Define $g : \omega \times X \to \tau$ as follow: $g(n, x) = xV_n$ for each $n \in \omega$ and $x \in X$. It is obvious that $\{g(n, x)\colon n \in \omega\}$ is a local base at x. Suppose $y \in g(n+1, x)$, then $y \in xV_{n+1}$, $y = xv_1$ for some $v_1 \in V_{n+1}$. Take $z \in g(n+1, y)$, then $z = yv_2$ for some $v_2 \in V_{n+1}$. $z = yv_2 = xv_1v_2 \in xV_{n+1}V_{n+1} \subset xV_n = g(n, x)$. By Lemma 2.1, X is quasi-metrizable. $\square$

Question 2.1. Is a first-countable paratopological group non-Archimedean quasi-metrizable?

Recall some generalized metrizable spaces. Let (X, τ) be a topological space. A function $g : \omega \times X \to \tau$ satisfies that $x \in g(n, x)$ for each $x \in X$, $n \in \omega$. A space X is a *β-space* (Hodel, [12, Definition 7.7]) if there is a function $g : \omega \times X \to \tau$ such that if $x \in g(n, x_n)$ for each $n \in \omega$, then the sequence $\{x_n\}$ has a cluster point in X. A space X is a *γ-space* (Hodel, [12, Definition 10.5]) if there exists a function $g : \omega \times X \to \tau$ such that (i) $\{g(n, x)\colon n \in \omega\}$ is a local base at $x \in X$; (ii) for each $n \in \omega$ and $x \in X$, there exists an $m \in \omega$ such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$.

The β-spaces are quite general [12]: Among Hausdorff spaces, all $w\Delta$-spaces, semi-stratifiable spaces, Σ-spaces are β-spaces.

Every quasi-metrizable space is a γ-space. Hodel [12, Theorem 10.7] proved that if a Hausdorff space X is a β-space and a γ-space, then X is developable.[2] So we have the following.

[2] A space X is *developable* [12] if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\{\mathrm{st}(x, \mathcal{U}_n)\colon n \in \omega\}$ forms a local base at x for every $x \in X$, where $\mathrm{st}(x, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n\colon x \in U\}$.

Corollary 2.1. *If G is a first-countable Hausdorff paratopological group and a β-space, then G is developable.*

Remark. The condition 'β-space' is essential. Sorgenfrey line [11, Example 1.2.2] is a non-developable first-countable paratopological group. A p-space need not be a β-space, the authors do not know the following.

Question 2.2. Let G be a first-countable paratopological group, if G is a p-space, is G developable?

Since among submetacompact spaces, a p-space is equivalent to a $w\Delta$-space, hence a submetacompact p-space is a β-space. We may ask the following question:

Question 2.3. Is every regular T_1, first-countable paratopological group submetacompact?

A space X is said to be *weakly first-countable* [1] if each point in X has a countable weak-base. Nedev and Choban [22] proved that every weakly first-countable topological group is metrizable, and Nyikos [24] proved the following.

Lemma 2.2. *([6, Theorem 4.7.5]) Every weakly first-countable Hausdorff paratopological group is first-countable.*

A function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a *symmetric* on the set X if, for each $x, y \in X$, (i) $d(x, y) = 0 \Leftrightarrow x = y$; (ii) $d(x, y) = d(y, x)$. A space X is said to be *symmetrizable* if there is a symmetric d on X satisfying the following condition: $U \subset X$ is open if and only if for each $x \in U$, there exists $\varepsilon > 0$ with $B(x, \varepsilon) = \{y \in X\colon d(x, y) < \varepsilon\} \subset U$.

Corollary 2.2. *([16, Theorem 2.2]) If G is a symmetrizable paratopological group, then G is developable.*

Proof. Since every symmetrizable space is weakly first-countable, G is first-countable by Lemma 2.2. Since every first-countable symmetrizable space is a β-space [12, Theorems 9.6 and 7.8], G is developable by Corollary 2.1. □

Arhangel'skii [2] proved that a bisequential topological group is metrizable. It is natural to ask the following.

Question 2.4. Is a regular T_1, bisequential paratopological group first-countable?

In fact, the authors even don't know if a countable, bisequential paratopological group is first-countable.

Every symmetrizable paratopological group is developable, hence a p-space. Bouziad [8] proved that a regular T_1, Baire semitopological group that is a p-space is a topological group. Then we have the following corollary.

Corollary 2.3 *(Arhangel'skii and Reznichenko). ([5]) Every regular T_1, Baire symmetrizable paratopological group is a metrizable topological group.*

Example 2.1. There exists a separable, developable, Hausdorff paratopological group that is not metrizable.

Proof. Let $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$ be real numbers, rational numbers groups with usual addition respectively. Let $(G, +) = (\mathbb{R}, +) \times (\mathbb{Q}, +)$, and define

$$(a_1, r_1) + (a_2, r_2) = (a_1 + a_2, r_1 + r_2), \quad \text{for each } (a_1, r_1), (a_2, r_2) \in (G, +).$$

Then $(G, +)$ is a Abelian group. Define a neighborhood base of $(a, r) \in G$ as follow:

$$\mathcal{B}_{(a,r)} = \big\{\big\{(a, r)\big\} \cup \big((a - 1/n, a + 1/n) \times (r, r + 1/n)\big)\colon n \in \mathbb{N}\big\}.$$

G is a topological space with topology generated by $\bigcup\{\mathcal{B}_{(a,r)}\colon (a, r) \in G\}$. It is easy to see that $(G, +)$ is a Hausdorff space. Since

$$\begin{aligned}&\big\{(a, r_1)\big\} \cup (a - 1/4n, a + 1/4n) \times (r_1, r_1 + 1/4n) + \big\{(b, r_2)\big\} \cup (b - 1/4n, b + 1/4n) \times (r_2, r_2 + 1/4n)\\ &\quad \subset \big\{(a + b, r_1 + r_2)\big\} \cup (a + b - 1/n, a + b + 1/n) \times (r_1 + r_2, r_1 + r_2 + 1/n),\end{aligned}$$

the operation '+' is jointly continuous, hence $(G, +)$ is a first-countable paratopological group. G is also separable since $\mathbb{Q} \times \mathbb{Q}$ is a countable dense subset of G. Fix $r \in \mathbb{Q}$, it is easy to see that $\{(a, r)\colon a \in \mathbb{R}\}$ is closed discrete (uncountable) subset of G, then G is not metrizable.

Since $G = \bigcup_{r \in \mathbb{Q}}\{(a, r)\colon a \in \mathbb{R}\}$, then G has a σ-discrete network, hence it is a σ-space, therefore it is a β-space [12, Theorem 7.8]. By Corollary 2.1, G is a developable space. □

Remark. Example 2.1 gives a partial answer to [25, Question 3.2]: Is every Moore paratopological group metrizable? Example 2.1 also shows that a first-countable paratopological groups need not be meta-Lindelöf.

Definition 2.1 *(Borges).* ([12, Definition 5.6]) A space is called *stratifiable* if there is a function G which assigns to each $n \in \omega$ and closed set $H \subset X$, an open set $G(n, H)$ containing H such that

(i) $H = \bigcap_{n\in\omega} G(n, H)$;
(ii) $H \subset K$ implies $G(n, H) \subset G(n, K)$;
(iii) $H = \bigcap_{n\in\omega} \overline{G(n, H)}$.

Lemma 2.3. *([12, Theorem 5.8]) A regular T_1 space (X, τ) is stratifiable if and only if there exists a function $g : \omega \times X \to \tau$ such that* (i) $\{x\} = \bigcap_{n\in\omega} g(n, x)$; (ii) *if $y \in g(n, x_n)$, then $x_n \to y$;* (iii) *if $y \notin H$, where H is closed, then $y \notin \bigcup\{g(n, x): x \in H\}$ for some $n \in \omega$.*

A symmetric $d : G \times G \to \mathbb{R}^+ \cup \{0\}$ of a paratopological group G is called *left-invariant* if for any $a, x, y \in G$, $d(x, y) = d(ax, ay)$.

Theorem 2.1. *Let G be a regular T_1 paratopological group with a left-invariant symmetric. Then G is metrizable.*

Proof. Let $d : G \times G \to \mathbb{R}^+ \cup \{0\}$ be a left-invariant symmetric, and let e be the neutral element of G. Since G is weakly first-countable, by Lemma 2.2, G is first-countable, hence G is quasi-metrizable by Proposition 2.1.

Now we prove that G is a stratifiable space. Put

$$B\left(x, 1/2^n\right) = \left\{y \in G: d(x, y) < 1/2^n\right\},$$

and fix a local base $\{V_n: n \in \omega\}$ at e such that $V_n \subset \operatorname{int}(B(e, 1/2^{k_n}))$, $V_{n+1}^2 \subset V_n$ and $d(e, x) > 1/2^{k_{n+1}}$ if $x \notin V_n$, where $n < k_n < k_{n+1}$. Define

$$g(n, x) = xV_n, \quad \text{for each } n \in \omega,\ x \in G.$$

Obviously, (i) of Lemma 2.3 is satisfied. For each $n \in \omega$, if $y \in g(n, x_n) = x_nV_n$, then $x_n^{-1}y \in V_n$, so

$$d(x_n, y) = d\left(x_n^{-1}x_n, x_n^{-1}y\right) = d\left(e, x_n^{-1}y\right) < 1/2^{k_n},$$

hence $x_n \to y$. (ii) of Lemma 2.3 is satisfied. Let H be a closed subset of G, $y \notin H$, then $yV_n \cap H = \emptyset$ for some $n \in \omega$.

Claim. $yV_{n+2} \cap g(n + 2, x) = \emptyset$ for each $x \in H$.

Suppose not, let $z \in yV_{n+2} \cap xV_{n+2}$, then $x^{-1}z \in V_{n+2}$, $d(z, x) = d(x^{-1}z, e) < 1/2^{k_{n+2}}$. $z^{-1}x \in V_{n+1}$, otherwise $d(z^{-1}x, e) > 1/2^{k_{n+2}}$. Since $z^{-1}x \in V_{n+1}$, $x \in zV_{n+1} \subset yV_{n+2}V_{n+1} \subset yV_n$, this is a contradiction with $yV_n \cap H = \emptyset$.

$yV_{n+2} \cap (\bigcup\{g(n+2, x): x \in H\}) = \emptyset$, then $y \notin \overline{\bigcup\{g(n + 2, x): x \in H\}}$. By Lemma 2.3, G is stratifiable, hence G is metrizable by [12, Corollary 10.8(ii)]. □

3. Topological groups with a point-countable covers

All spaces in this section are regular T_1 unless stated otherwise.

A cover $\mathcal{P}$ of a topological space X is *point-countable* (*point-finite*) if every point of X belongs to at most countably many (finitely many) elements of $\mathcal{P}$. For a cover $\mathcal{P}$ of a space X we say that X is *determined by* $\mathcal{P}$ [13] provided that a set $F \subset X$ is closed in X if and only if its intersection $F \cap P$ with every $P \in \mathcal{P}$ is closed in P.

Let $\mathcal{P}$ be a family of subsets of a space X. Then $\mathcal{P}$ is a *cs-network at a point* $x \in X$ if whenever $\{x_n\}$ is a sequence converging to x and U is a neighborhood of x, there are $k \in \omega$ and $P \in \mathcal{P}$ such that $\{x\} \cup \{x_n: n \geqslant k\} \subset P \subset U$. Similarly, $\mathcal{P}$ is a *cs*-network at a point* $x \in X$ if whenever $\{x_n\}$ is a sequence converging to x and U is a neighborhood of x, there are an infinite $A \subset \omega$ and $P \in \mathcal{P}$ such that $\{x\} \cup \{x_n: n \in A\} \subset P \subset U$.

Lemma 3.1. *Let X be a space determined by a point-countable cover $\mathcal{P}$ consisting of first-countable subsets. Then X is a sequential spaces with a countable cs-network at each point in X.*

Proof. Let Z be the disjoint topological sum $\bigoplus\mathcal{P}$ of $\mathcal{P}$, and $f : Z \to X$ be the obvious map. Then f is a quotient map by [13, Lemma 1.8]. Since Z is first-countable, X is a sequential space. Fix a point $x \in X$, and let

$$\mathcal{P}_x = \{P \in \mathcal{P}: x \in P\} = \{P_n: n \in \omega\}.$$

Each P_n is first-countable, let $\{V(n, i): i \in \omega\}$ be countable local base at x in P_n, then $\{V(n, i): n, i \in \omega\}$ is a countable cs^*-network at x. In fact, let $\{x_k\}$ be a sequence converging to $x \in U$ with U open in X, then there is a sequence L converging to some point z in Z such that $f(L)$ is a sequence of $\{x_k\}$ by [27, Lemma 1.6]. Since $z \in f^{-1}(x) \subset f^{-1}(U)$, then $z = x \in P_n$ in Z for some $n \in \omega$, and $V(n,m) \subset f^{-1}(U) \cap P_n$ in Z for some $m \in \omega$. We can assume that $L \subset V(n,m)$ in Z, then $\{x\} \cup f(L) \subset V(n,m) \subset U$ in X. Hence $\{V(n,i)\colon n, i \in \omega\}$ is a countable cs^*-network at x. By [26, Lemma 2.2], X has a countable cs-network at every $x \in X$. $\square$

The S_κ is the quotient space obtained from the disjoint sum of κ many convergent sequences vis identifying limit points of all these sequences. S_ω is called *sequential fan*.

A topological space X is called an *α_4-space* (*α_7-space*), if for any sequence $S_n \subset X$ ($n \in \omega$), converging to a point $x \in X$ there is a sequence $S \subset X$ converging to x (some point $y \in X$) and such that $S_n \cap S \neq \emptyset$ for infinitely many sequences S_n.

Lemma 3.2. *If a sequential space X has no closed copy of S_ω, then X is an α_7-space.*

Proof. Let $\{S_n\colon n \in \omega\}$ be a collection of convergent sequences in X with $S_n \to x \in X$ for each $n \in \omega$. Put $Z = \{x\} \cup (\bigcup\{S_n\colon n \in \omega\})$.

If Z is not closed in X, there is a sequence $S \subset Z$ such that S converges to some point $y \in X \setminus Z$ because X is a sequential space. Then $S_n \cap S \neq \emptyset$ for infinitely many sequences S_n. If Z is closed in X, then it is not a copy of S_ω. There exists $x_k \in S_{n_k}$ ($k \in \omega$) such that $\{x_k\colon k \in \omega\}$ is not closed in Z. Thus there is a convergent sequence $S \subset \{x_k\colon k \in \omega\}$. Hence X is an α_7-space. $\square$

Theorem 3.1. *Let G be a paratopological group determined by a point-countable cover consisting of first-countable subsets. Then G is first-countable if it contains no closed copy of S_ω.*

Proof. By Lemmas 3.1 and 3.2, G is an α_7-space. Then G has countable sb-character[3] by the same proof of [7, Lemma 3]. It is easy to check that a sequential space having countable sb-character is weakly first-countable, then G is first-countable by Lemma 2.2. $\square$

Corollary 3.1. *Let G be a topological group determined by a point-countable cover consisting of first-countable subsets. Then G is metrizable if it contains no closed copy of S_ω.*

Remark. Corollary 3.1 gives a partial answer to Nogura–Shakhmatov–Tanaka's question [23, Questions 3.9 and 3.10].

Corollary 3.2. *A paratopological group G determined by a point-finite cover consisting of first-countable subspaces is first-countable.*

Proof. By [23, Lemma 2.7], G is an α_4-space. Then G contains no closed copy of S_ω, hence G is first-countable by Theorem 3.1. $\square$

The following lemma is a easy modification of Lemma 4 in [7].

Lemma 3.3. *Let $(G,*)$ be a sequential topological group. Then G contains no closed copy of S_ω or every first-countable closed subset of G is locally countably compact.*

Proof. Suppose not, $(G,*)$ contains a closed copy of $S_\omega = \{e\} \cup \{x_{n,m}\colon n, m \in \omega\}$, where $x_{n,m} \to e$ as $m \to \infty$ for each $n \in \omega$. $(G,*)$ also contains a closed first-countable subset H that is not locally countably compact. Without loss of generality, we assume $e \in H$ and e has a countable decreasing local base $\{V_n\colon n \in \omega\}$ at e in H, each V_n is not locally countably compact. For each $n \in \omega$, let $\{y_{n,m}\colon m \in \omega\} \subset V_n$ be a closed discrete subset of H. Fix n, it is easy to see that $D_n = \{x_{n,m} * y_{n,m}\colon m \in \omega\}$ is closed and discrete in $(G,*)$. Hence there exists $k_n \in \omega$ such that $e \neq x_{n,m} * y_{n,m}$ for all $m > k_n$. So we may assume that $e \notin \{x_{n,m} * y_{n,m}\colon m \in \omega\}$. Let $A = \{x_{n,m} * y_{n,m}\colon n, m \in \omega\}$, then $e \in \overline{A \setminus \{e\}}$. A is not closed, since $(G,*)$ is sequential, there is a sequence $S \subset A$ convergent to a point $a \notin A$. D_n is closed and discrete, then $S \cap D_n$ is finite. Consequently, there is a subsequence $\{x_{n_i,m_i} * y_{n_i,m_i}\colon i \in \omega\} \subset S$ with $n_{i+1} > n_i$. Since $y_{n_i,m_i}^{-1} \to e$, $x_{n_i,m_i} * y_{n_i,m_i} * y_{n_i,m_i}^{-1} \to a * e = a$, i.e. $x_{n_i,m_i} \to a$. This is a contradiction since $\{x_{n_i,m_i}\colon i \in \omega\}$ is discrete. $\square$

Recall the concept of k-networks. A collection $\mathcal{P}$ of subsets of a space X is a *k-network* [12] if whenever K is a compact subset of an open set U in X, there exists a finite $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$.

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[3] A space X is called to have countable *sb-character* if for every $x \in X$, there is a countable network $\mathcal{P}$ at x consisting of the sequential neighborhoods of x, that is, every sequence converging to x is eventually in each element of $\mathcal{P}$.

Remark. If a space X is determined by a point-countable cover $\mathcal{P}$ consisting of metric subsets. Then the space Z in the proof of Lemma 3.1 is metrizable and the obvious map $f : Z \to X$ is a quotient s-map, hence X has a point-countable k-network by [13, Theorem 6.1].

Lemma 3.4. *Let X a k-space with a point-countable k-network, in which every closed first-countable subset is locally countably compact. Then X has a point-countable k-network consisting of cosmic subsets.*

Proof. Since every closed first-countable subset in X is locally countably compact, X has a point-countable k-network $\mathcal{P}$ such that $\overline{P}$ is countably compact in X for each $P \in \mathcal{P}$ by [15, Theorem 2.3]. For each $P \in \mathcal{P}$, $\overline{P}$ is compact metrizable by [13, Theorem 4.1], then P is a cosmic subspace[4] in X.

Lemma 3.5. *([18, Corollary 2.7]) Let G be a sequential topological groups with a point-countable k-network consisting of cosmic subsets. Then G has an open subgroup which is cosmic.*

Theorem 3.6. *Let G be a sequential topological group with a point-countable k-network. Then G is a metrizable space or a topological sum of cosmic spaces.*

Proof. By Lemma 3.3, we consider following two cases:

Case 1. G contains no closed copy of S_ω.

Since a k-space that has a point-countable k-network and contains no closed copy of S_ω is weakly first-countable [14, Theorem 3.13 and Corollary 3.9], hence G is metrizable by Lemma 2.2.

Case 2. Every closed first-countable subset is locally countably compact.

By Lemmas 3.4 and 3.5, G is a topological sum of cosmic subspaces. □

Corollary 3.3. *A sequential topological group with a point-countable k-network is paracompact.*

4. Symmetrizable semitopological groups

In [10], Cao, Drozowski and Piotrowski asked if every symmetrizable semitopological group is a Moore space. They further asked "Must every symmetrizable Hausdorff Baire semitopological group be a topological group?" The answer is "No". In fact, a symmetrizable semitopological group need not be first-countable.

Example 4.1. There is a separable, symmetrizable, Hausdorff semitopological group that is not first-countable.

Proof. Let $G = \mathbb{R}^2$ with usual addition "+", then $(G, +)$ is a group. Define $d : G \times G \to \mathbb{R}^+ \cup \{0\}$ as follow:

$$d\big((x, y), (x', y')\big) = \begin{cases} |x - x'|, & x \neq x',\ y = y'; \\ |y - y'|, & x = x',\ y \neq y'; \\ 0, & x = x',\ y = y'; \\ 1, & \text{otherwise}. \end{cases}$$

It is easy to check that G is Hausdorff and d is a symmetric on $(G, +)$. Endow $(G, +)$ with the topology generated by d, then $(G, +)$ is a semitopological group. $\mathbb{Q} \times \mathbb{Q} \subset (G, +)$ is a countable dense subset of G. In fact, let V be an open subset of G. Pick $x = (x', x'') \in V$, there exists $n \in \mathbb{N}$ such that $\{y \in G\colon d(x, y) < 1/n\} \subset V$, we find $r' \in \mathbb{Q}$ with $|r' - x'| < 1/n$, then $z = (r', x'') \in V$. There exists $k \in \mathbb{N}$ such that $\{y \in G\colon d(z, y) < 1/k\} \subset V$, find $r'' \in \mathbb{Q}$ with $|r'' - x''| < 1/k$, then $(r', r'') \in V$, therefore G is separable. But G is not first-countable since no sequence in $\mathbb{Q} \times \mathbb{Q}$ converges to (p', p''), where p', p'' are irrational numbers. □

A space X is said to be *semi-metric* [12] if there is a symmetric d on X such that for each $x \in X$, $\{B(x, \varepsilon)\colon \varepsilon > 0\}$ forms a neighborhood base at x.

Example 4.2. There is a separable, Baire, semi-metric, semitopological group that is not a paratopological group.

[4] A space is called a *cosmic space* if it has a countable network.

Proof. Let $G = (\mathbb{R}^2, +)$ with the 'bowtie' topology [12]; that is, a neighborhood $U((s,t), \varepsilon, \delta)$ of a point $(s,t) \in G$ is the 'bowtie':

$$\{(s,t)\} \cup \{(s',t'):\ 0 < |s - s'| < \varepsilon,\ |(t'-t)/(s'-s)| < \delta\},$$

where $\varepsilon > 0$ and $\delta > 0$ can vary.

Define

$$d((s,t),(s',t')) = \begin{cases} 0, & s = s',\ t = t'; \\ 1, & s = s',\ t \neq t'; \\ |s - s'| + |(t - t')/(s - s')|, & \text{otherwise.} \end{cases}$$

It is easy to check that G is regular T_1, d is semi-metric and G is a separable, semitopological group.

We prove that G is a Baire space. Let $\mathbb{R}^2$ be the real plane with usual topology.

Let $\{V_n:\ n \in \mathbb{N}\}$ be a sequence of open dense subsets of G. Fix $n \in \mathbb{N}$, for $(s,t) \in V_n$, $U((s,t), 1/m, 1/m) \subset V_n$ for some $m \in \mathbb{N}$, let

$$W_n(s,t) = U((s,t), 1/m, 1/m) \setminus \{(s,t)\}, \qquad W_n = \bigcup\{W_n(s,t):\ (s,t) \in V_n\}.$$

Since $W_n(s,t)$ is open in $\mathbb{R}^2$ and V_n is dense in G, W_n is an open dense subset in $\mathbb{R}^2$. It is well known that $\mathbb{R}^2$ has Baire property, then $\bigcap_{n\in\mathbb{N}} W_n (\subset \bigcap_{n\in\mathbb{N}} V_n)$ is dense in $\mathbb{R}^2$. Since every open subset in G contains an open subset in $\mathbb{R}^2$, $\bigcap_{n\in\mathbb{N}} W_n$ is dense in G, hence $\bigcap_{n\in\mathbb{N}} V_n$ is dense in G.

G is not a paratopological group.

Suppose not, then G is a Moore space by Corollary 2.2, hence a p-space. By Bouziad's result [8], G is a topological group, hence a separable metric space. But it is easy to see $\{(0,t):\ t \in \mathbb{R}\}$ is discrete, this is a contradiction. □

5. Remainders of (para-)topological groups

All spaces in this section are regular T_1 unless stated otherwise.

In this section, we discuss the remainders of topological groups and paratopological groups in Hausdorff compactifications.

By a *remainder* of a Tychonoff space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X.

A space X is said to have a *regular G_δ-diagonal* if the diagonal $\Delta = \{(x,x):\ x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$. According to Zenor [28], a space X has a regular G_δ-diagonal if and only if there exists a sequence $\{\mathcal{G}_n:\ n \in \omega\}$ of open covers of X with the following property:

(*) For any two distinct point y and z in X, there are open neighborhoods O_y and O_z of y and z, respectively, and $k \in \omega$ such that no element of $\mathcal{G}_k$ intersects both O_x and O_y.

Arhangel'skii [3] proved that if a remainder of a non-locally compact topological group in a Hausdorff compactification has a G_δ-diagonal, then G is separable and metrizable. This is not true when G is a paratopological group. In fact, Alexandorff's double-arrow space is a Hausdorff compactification of Sorgenfrey line, its remainder is still a copy of Sorgenfrey line, so the remainder has a regular G_δ-diagonal [16], but Sorgenfrey line is not metrizable. We may ask the following question:

Question 5.1. Let G be a non-locally compact paratopological group. Suppose the remainder $Y = bG \setminus G$ has a regular G_δ-diagonal, does G have a regular G_δ-diagonal?

The following theorem gives a partial answer to the above question.

Theorem 5.1. *Let G be a non-locally compact Abelian paratopological group in which every compact subset is first-countable. Suppose the remainder $Y = bG \setminus G$ has a regular G_δ-diagonal, then G has a regular G_δ-diagonal.*

Proof. We consider following two cases:

Case 1. Y is pseudocompact.

Since Y has a regular G_δ-diagonal, the Y is metrizable, hence Y is compact. It means that G is locally compact. This is a contradiction.

Case 2. Y is not pseudocompact.

By [4, Lemma 2.1], there exists a non-empty compact subspaces F of G which has a countable strong π-base[5] $\{V_n: n \in \mathbb{N}\}$ in G, we may assume $V_{n+1} \subset V_n$ (in fact, we may consider $\{\bigcup_{n \geqslant k} V_n: k \in \mathbb{N}\}$).

Claim. *There exists $x \in F$ such that any neighborhood of x in G meets every V_n.*

Suppose not, for each $x \in F$, there is an neighborhood $V(x)$ of x in G and $n(x) \in \mathbb{N}$ such that $V(x) \cap V_n = \emptyset$ for $n > n(x)$. $F \subset \cup\{V(x_i): i \leqslant k\}$ for some k. Let $n_0 = \max\{n(x_i): i \leqslant k\}$, $V_n \cap (\bigcup\{V(x_i): i \leqslant k\}) = \emptyset$ for each $n > n_0$, this is a contradiction.

Fix $x \in F$ such that any neighborhood of x meets every V_n. F is first-countable, let $\{W_n: n \in \mathbb{N}\}$ be a decreasing base at x in F. $W_n = F \cap U_n$, where U_n is open in G, we may assume $U_{n+1} \subset U_n$. Let $R_n = V_n \cap U_n$ for $n \in \mathbb{N}$, $\{R_n: n \in \mathbb{N}\}$ is a decreasing strong π-base converging to x. Without loss of generality, we assume $x = e$.

Let $\mathcal{G}_n = \{xR_n: x \in G\}$ for each n, then $\{\mathcal{G}_n: n \in \mathbb{N}\}$ is a sequence of open coverings of G. By Zenor's characterization [28] of regular G_δ-diagonal, we only prove the following: For $y, z \in G$, $y \neq z$, there is $k \in \mathbb{N}$ such that no element of $\mathcal{G}_k$ intersects both $y'R_k$ and $z'R_k$, where $y \in y'R_k$, $z \in z'R_k$.

Suppose not; for any $n \in \mathbb{N}$, there is an element $x_nR_n \in \mathcal{G}_n$ and $y_n, z_n \in G$ such that $y \in y_nR_n, z \in z_nR_n$, $y_nR_n \cap x_nR_n \neq \emptyset$ and $z_nR_n \cap x_nR_n \neq \emptyset$. Then there exist $a_n, b_n, c_n, d_n, u_n, v_n \in R_n$ such that

$$y = y_nu_n, \quad z = z_nv_n \quad \text{and} \quad y_na_n = x_nb_n, \quad x_nc_n = z_nd_n.$$

Then $x_n = b_n^{-1}a_ny_n = b_n^{-1}a_nu_n^{-1}y$. On the other hand, $x_n = c_n^{-1}z_nd_n = c_n^{-1}d_nv_n^{-1}z$. Then $b_n^{-1}a_nu_n^{-1}y = c_n^{-1}d_nv_n^{-1}z$, hence $yz^{-1} = a_n^{-1}c_n^{-1}v_n^{-1}b_nu_nd_n$. Since $a_nc_nv_n \to e$ $(n \to \infty)$, then $a_nc_nv_na_n^{-1}c_n^{-1}v_n^{-1}b_nu_nd_n \to yz^{-1}$, $b_nu_nd_n \to yz^{-1}$, but $b_nu_nd_n \to e$, that means $yz^{-1} = e$, therefore $y = z$. this is a contradiction.

G has a regular G_δ-diagonal. $\square$

Lemma 5.1. *([9]) Suppose X is of countable tightness and $A \subset X$. If $\mathcal{P}$ is any point-countable collection of X, then there are at most countably many minimal finite subcollections $\mathcal{F} \subset \mathcal{P}$ such that $A \subset (\bigcup \mathcal{F})^\circ$.*

Let X be a space and $x \in X$. A collection of nonempty open sets $\mathcal{U}$ of X is called a *π-base at x* if for every open set O with $x \in O$, there exists an $U \in \mathcal{U}$ such that $U \subset O$. The idea of the following lemma bases on the proof of [13, Proposition 3.2].

Lemma 5.2. *Let X be a pseudo-open s-image of a space with a point-countable base. Then X has a countable π-base at each point.*

Proof. Since X is a pseudo-open s-image of a space with a point-countable base, by [13, Propositions 6.2 and 6.3(a)], X is a pseudo-open s-image of a metric space. Tanaka [27] characterized a pseudo-open s-images of a metric spaces as a Fréchet space with a point-countable cs^*-network. Let $\mathcal{P}$ be a point-countable cs^*-network of X.

Fix $x \in X$, if x is an isolated point, the X has a countable π-base at x. If x is not an isolated point, there is a non-trivial sequence $C_0 = \{x_n: n \in \mathbb{N}\}$ converging to x. We may assume that each x_n is not an isolated point. Otherwise $\{\{x_n\}: x_n$ is an isolated point$\}$ is a countable π-base at x.

Put

$$\mathcal{F} = \{F: F \subset C_0,\ |F| < \omega\}, \qquad \mathcal{E} = \{C_0 \setminus F: F \in \mathcal{F}\},$$
$$\mathcal{G}_E = \left\{\left(\bigcup \mathcal{P}'\right)^\circ: E \subset \left(\bigcup \mathcal{P}'\right)^\circ,\ \mathcal{P}' \text{ is a minimal finite subcollection of } \mathcal{P}\right\}.$$

By Lemma 5.1, $\mathcal{G}_E$ is countable.

Let $\mathcal{G} = \{\mathcal{G}_E: E \in \mathcal{E}\}$.

Claim. $\mathcal{G}$ is a π-base at x.

Suppose not, there is an open neighborhood V of x such that V contains no element of $\mathcal{G}$. Without loss of generality, we assume $\{x_n: n \in \mathbb{N}\} \subset V$. Let $\mathcal{Q} = \{P \in \mathcal{P}: P \subset V\}$. Then there is no finite subcollection $\mathcal{P}'' \subset \mathcal{Q}$ such that $(\bigcup \mathcal{P}'')^\circ$ contains any element of $\mathcal{E}$.

Let $\mathcal{Q}(A) = \{P \in \mathcal{Q}: P \cap A \neq \emptyset\}$ for $A \subset X$. Since $\mathcal{Q}$ is point-countable, we write $\mathcal{Q}(C_0) = \{P_{n,0}: n \in \mathbb{N}\}$.

$P_{1,0}^\circ$ does not contain any element of $\mathcal{E}$, there exists $x_{n_1} \notin P_{1,0}^\circ$. Since X is Fréchet, there is a sequence $C_1 \subset V \setminus (P_{1,0} \cup \{x\})$ converging to x_{n_1}. We write $\mathcal{Q}(C_1) = \{P_{n,1}: n \in \mathbb{N}\}$. $(\bigcup\{P_{i,j}: i \leqslant 2,\ j < 2\})^\circ$ does not contain any element of $\mathcal{E}$, there exists $x_{n_2} \notin (\bigcup\{P_{i,j}: i \leqslant 2, j < 2\})^\circ$ with $n_2 > n_1$. Then there is a sequence $C_2 \subset V \setminus ((\bigcup\{P_{i,j}: i \leqslant 2, j < 2\}) \cup \{x\})$. By induction, we can choose countable sequences C_i, x_{n_i} $(i \in \mathbb{N})$ such that $n_i < n_j$ if $i < j$, $x \notin C_i$ and $C_n \cap P_{i,j} = \emptyset$ for $i \leqslant n,\ j < n$. The

[5] A *strong π-base* of a space X at a subset F of X is an infinite family γ of non-empty open subsets of X such that every open neighborhood of F contains all but finitely many elements of γ.

last condition implies that no $P \in \mathcal{Q}$ meets infinitely many C_i. Put $C = \bigcup\{C_i: i \in \omega\}$, $x \in \overline{C}$, then there is a sequence $K \subset V$ converging to x. Since $\mathcal{P}$ is a cs^*-network, there is a $P \in \mathcal{P}$ such that $x \in P \subset V$ and P contains infinitely many elements of K, hence $P \in \mathcal{Q}$ and P meets infinitely many C_i, this is a contradiction. $\square$

It was proven that [17, Theorem 4] a non-locally compact topological group G and its Hausdorff compactification bG are separable and metrizable if its remainder is a quotient s-image of a space with a point-countable base and has countable π-base at each point. By applying Lemma 5.2, we have the following.

Theorem 5.2. *Let G be a non-locally compact topological group, if the remainder $Y = bG \setminus G$ of a Hausdorff compactification of G is a pseudo-open s-image of a space with a point-countable base, then G and bG are separable and metrizable.*

Question 5.2. Let G be a non-locally compact topological group, if the remainder $Y = bG \setminus G$ of a Hausdorff compactification of G has a point-countable weak base, are G and bG separable and metrizable?

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